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1988 J. Phys. A: Math. Gen. 21 L95

(http://iopscience.iop.org/0305-4470/21/2/006)

J. Phys. A: Math. Gen. 21 (1988) L95–L98. Printed in the UK

LETTER TO THE EDITOR

Berry's phase as the asymptotic limit of an exact evolution: an example

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Received 17 August 1987

Abstract. We exhibit, for arbitrary time variations of the parameters of the generalised harmonic oscillator Hamiltonian, a canonical transformation which facilitates an *exact* analysis of quantal phase and classical angle. Formulae for Berry's phase and Hannay's angle are readily obtained by taking the adiabatic limit of the exact theory.

According to the quantal adiabatic theorem [1, 2] a bound system with a non-degenerate spectrum evolves by clinging to the eigenstates of its instantaneous Hamiltonian when the parameters describing the system are changed sufficiently slowly. Consequently, the overall phase of the wavefunction changes by an amount proportional to the time integral of the instantaneous parameter-dependent energy. An important modification of this theorem discovered by Berry [3] is that, for circuital adiabatic excursions in parameter space, the wavefunction of the system picks up an additional non-integrable phase [3, 4] which depends on the geometry of the circuit and, of course, on the eigenstate under consideration. This apparently innocuous phase has far-reaching consequences and its inclusion promises a better understanding of some of the outstanding enigmas of quantum physics. Berry's phase has found application in molecular dynamics in the Born–Oppenheimer approximation [5–7] and has recently been used by Ham [8] to elucidate the order of the lowest vibronic states in the dynamic Jahn–Teller effect for defects in crystals. The physics of the quantum Hall effect [9–11] is another area in which the phase turns up, as it does in the case of the quantum field theory for Hamiltonians that develop anomalies [12–15]. A recent experiment by Tomita and Chiao [16–18] provides a direct confirmation of the existence of this phase, as does the early optical experiment of Pancharatnam [19].

Berry's phase has a, perhaps less well known, classical counterpart: Hannay's angle [20, 21]. An integrable system evolves, according to the classical adiabatic theorem [22], by clinging to its instantaneous invariant tori when transported slowly in parameter space. Hannay [20] found that, under closed adiabatic cycling, the system suffers an extra shift in the angle variable in addition to that dictated by the theorem. This extra angle change depends on the geometry of the parameter space circuit and on the conserved tori actions. A semiclassical connection [21] that has been shown to exist between this angle and Berry's phase asserts, *inter alia*, that a classical system exhibiting Hannay's angle must feature Berry's phase at the quantal level.

To date, formulae for the Berry phase and the Hannay angles have been derived for a number of systems. We are concerned here with one such system: the so-called generalised harmonic oscillator (GHO). It is our purpose to demonstrate that an *exact* analysis, valid for any time variation of the parameters, can be carried out for this simple yet physically important example. Working with this particular example it is not necessary to invoke the adiabatic technique of averaging over fast frequencies as was done, for example, in Berry's derivation [21] of the general formula for Hannay's angles. In fact, the Berry phase and Hannay angle will be found to be contained within our more general and exact analysis, which can also be used to calculate corrections to the adiabatic results for the GHO.

Turning to technical matters, we construct a time-dependent canonical transformation whereby the standard form of the GHO Hamiltonian is taken into one which is a function of the new canonical momentum only. In this new frame the time evolution can be solved in terms of the exact time-dependent frequencies of the system. The analysis then enables us to extract the geometric phase and its corresponding angle by taking the limit of infinitely slow parameter variations. The new canonical transformation is a generalisation of that discovered by Lewis [23] and used by him to construct an exact invariant of the motion for the ordinary harmonic oscillator with a time-dependent frequency. Lewis, in turn, used the general theory of Kruskal [24].

Consider, then, the GHO Hamiltonian

$$H=\tfrac{1}{2}[Zp^2+2Yqp+Xq^2] \tag{1}$$

where $\{X(t),\ Y(t),\ Z(t)\}=\boldsymbol{R}$, are the time-dependent parameters. The instantaneous H has a frequency $\omega=(XZ-Y^2)^{1/2}$ (Lewis worked with the case $\{X=X(t),\ Y=0,\ Z=\text{constant}\}$). The GHO has the exact invariant

$$I=\tfrac{1}{2}\left\{\frac{q^2}{r^2}+\left[r\left(p+\frac{Yq}{Z}\right)-\frac{\dot{r}q}{Z}\right]^2\right\} \tag{2}$$

where an overdot indicates time differentiation and where the auxiliary variable $r(t)$ is any solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{Z}\right)-r\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Y}{Z}\right)+\frac{Y^2}{Z}+\frac{Z}{r^4}-X\right\}=0. \tag{3}$$

That $\dot{I}=0$ is easily established by using Hamilton's equations together with equation (3) for $r(t)$.

The proposed canonical transformation is achieved by choosing I as the new momentum coordinate (P) whose conjugate (Q) is given by

$$Q=-\tan^{-1}\left[\frac{r^2}{q}\left(p+\frac{Yq}{Z}\right)-\frac{r\dot{r}}{Z}\right]. \tag{4}$$

The $F_2(q,P,t)$-type generator [25] for the transformation satisfies the pair of differential equations

$$\frac{\partial F_2}{\partial q}=P \qquad \frac{\partial F_2}{\partial P}=Q$$

which upon integration yield

$$F_2=-(n+\tfrac{1}{2})\pi P\pm P\sin^{-1}\left(\frac{q}{r\sqrt{2P}}\right)\pm\frac{q}{2r}\left(2P-\frac{q^2}{r^2}\right)^{1/2}+\frac{q^2}{2Z}\left(\frac{\dot{r}}{r}-Y\right). \tag{5}$$

For the new Hamiltonian we find the simple form

$$K=H+\frac{\partial F_2}{\partial t}=\frac{Z}{r^2}P. \tag{6}$$

Thus K is cyclic in Q and we have obtained an exact transformation to action-angle type variables.

The time evolution of the angle variable is governed by the new time-dependent frequencies

$$\omega' = Z/r^2 \tag{7}$$

so that the change in angle, in the time T required for the parameter values to describe a complete cycle, is given by

$$\theta(T) = \int_0^T \mathrm{d}t \frac{Z(t)}{r^2(t)} \tag{8}$$

which is the exact formula promised. Of course, this expression arises from the dynamics. However, if we proceed to take the adiabatic limit, obtained by substituting into equation (8) the adiabatic solution to equation (3), we find that

$$\theta(T) = \int_0^T \mathrm{d}t\, \omega(t) - \tfrac{1}{2} \iint \mathrm{d}\boldsymbol{S} \cdot [\nabla_{\boldsymbol{R}}(Z/\omega) \times \nabla_{\boldsymbol{R}}(Y/Z)]. \tag{9}$$

The integral over the surface spanning the contour, now *adiabatically* traversed in parameter space, is the Hannay angle arising dynamically from the new frequency ω', with its intrinsic geometric nature made manifest.

To obtain Berry's phase, we notice that the action variable $I(=P)$ can be thrown into a quadratic Hermitian form by introducing creation and annihilation operators:

$$\begin{aligned} a &= \frac{1}{\sqrt{2\hbar}} \left\{ \frac{q}{r} + \mathrm{i}\left[r\left(p + \frac{Yq}{Z}\right) - \frac{\dot{r}q}{Z}\right)\right\} \\ a^\dagger &= \frac{1}{\sqrt{2\hbar}} \left\{ \frac{q}{r} - \mathrm{i}\left[r\left(p + \frac{Yq}{Z}\right) - \frac{\dot{r}q}{Z}\right]\right\}. \end{aligned} \tag{10}$$

The operators satisfy $[a(t), a^\dagger(t)] = 1$. This implies that the Hamiltonian K has the time-dependent equispaced eigenvalue spectrum

$$E_m = \frac{\hbar Z}{r^2}(n + \tfrac{1}{2}).$$

The total quantal phase, as obtained by integrating over the instantaneous new frequencies, is obviously

$$-\frac{1}{\hbar} \int_0^T \mathrm{d}t \frac{\hbar Z}{r^2}(n + \tfrac{1}{2}) \tag{11}$$

which as before reduces, in the adiabatic limit, to

$$-(n + \tfrac{1}{2}) \left(\int_0^T \mathrm{d}t\, \omega(t) - \tfrac{1}{2} \iint \mathrm{d}\boldsymbol{S} \cdot [\nabla_{\boldsymbol{R}}(Z/\omega) \times \nabla_{\boldsymbol{R}}(Y/Z)] \right). \tag{12}$$

The second term in (12) is the familiar geometric circuit phase.

Formulae (8) and (11) show how, for any time variation of the parameters, the quantum phase and classical angle can be exactly determined. In (Q, P)-space the instantaneous eigenstates or the tori are clung to—for arbitrary time variations of the parameters—and total phase or angle changes are obtained by integrating the instantaneous frequencies ω' over the time elapsed. Despite this generality, however, extremely interesting results are obtained when the parameter variations are restricted to be simultaneously adiabatic and cyclic, since only in this limit do the results manifestly depend on the geometry of the circuit via the flux of the curl of the parameter space vector field $[(Z/\omega)\nabla_R(Y/Z)]$ through any surface bounded by the circuit.

Finally, we briefly mention two interesting directions in which the present analysis can be extended. The first is to other exactly integrable time-dependent systems. Then, in principle one could choose, through a canonical transformation, the invariants as new action variables (Arnold's theorem) [22]. An example that comes to mind is the time-dependent cubic oscillator for which Leach [26] has constructed an invariant. Secondly, one could extend our analysis of the GHO and calculate corrections to the adiabatic limit. Berry [27] has recently looked at this question for general Hamiltonians in the quantal case. He has given a nice iterative prescription for calculating the corrections. It would be very interesting to compare the results obtained from these two quite distinct-looking procedures.

It is a pleasure to thank Professors Michael Berry and Binayak Dutta-Roy for encouragement and stimulating discussions. We also thank Professor Berry for showing us his recent unpublished work on corrections to the Berry phase.

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